PMM U.S.S.R.,Vol.54,No.3,pp.302-308,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

MOTION OF CONTROLLABLE MECHANICAL SYSTEMS WITH SERVO-CONSTRAINTS*

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Some special features in the dynamics of systems with servo-constraints /1/, due to the fact that such constraints are non-ideal and disengageable, are studied. A constructive method is proposed to justify the axiom of ideal constraints for kinematically controllable systems and the principle of reduction of conditional to real constraints. The method is based on the general theory of motion of systems with non-ideal constraints, as developed in /2, 3/ as it applies to systems with friction. Equations which enable one to stabilize motions relative to the manifold determined by the servo-constraints are developed and analysed.

Beghin's theory /1/ is developed and subjected to a critcal analysis in /4-7/. The special freatures of the analytical treatment of systems with servo-constraints and systems with conditional constraints were analysed in /8/. However, owing to the different treatment of the reaction forces in servo-constraints and the lack of a rigorous justification of the principle of reduction of conditional to real constraints, the question of whether there is a more intimate connection between the two theories remained open. The method proposed here enables one to establish conditions under which the theories of /1, 4/ are compatible and to describe the dynamics of systems with servo-constraints with due allowance for the parametric disengagement of such constraints /9/.

1. We consider a mechanical system whose state, taking into account ideal holonomic constraints of the first kind /1/, is determined by coordinates $q_i\ (i = 1, 2, \ldots, n)$. Suppose that the system is subjected to prescribed forces Q_i and that its motion is limited by compatible and independent constraints, some of which are geometric:

$$f_\alpha (q_j, t) = 0 \quad (f_\alpha \in C_2;\ \alpha = 1, 2, \ldots, a) \tag{1.1}$$

and some kinematic (not necessarily linear):

$$\varphi_\beta (q_i, q_i^{\boldsymbol\cdot}, t) = 0 \quad (\varphi_\beta \in C_1;\ \beta = 1, 2, \ldots, b) \tag{1.2}$$

The possible displacements permitted by the constraints are determined by $a + b$ independent relations /9/

$$\sum_{i=1}^n \frac{\partial f_\alpha}{\partial q_i} \delta q_i = 0, \quad \sum_{i=1}^n \frac{\partial \varphi_\beta}{\partial q_i^{\boldsymbol\cdot}} \delta q_i = 0$$

and the manifold of admissible states of the system may be expressed as

$$q_i = a_i (q_j, t), \quad q_i^{\boldsymbol\cdot} = b_i (q_j, p_s, t) \ (a_i \in C_2,\ b_i \in C_1) \tag{1.3}$$

where $q_j\ (j = 1, 2, \ldots, p)$ are independent Lagrangian coordinates and $p_s\ (s = 1, 2, \ldots, r)$ independent velocity parameters. Under these conditions the variations of the coordinates, δq_i, can be expressed in terms of arbitrary quantities $\delta \pi_s$ as follows:

$$\delta q_i = \sum_{s=1}^r \frac{\partial b_i}{\partial p_s} \delta \pi_s$$

Let us assume that the first c of the constraints (1.1) and the first d of the constraints (1.2) are of the first kind. Denoting the reaction forces of the constraints of the first kind by N_i and those of the servo-constraints by Φ_i, we can express the resultant reactions as $R_i = N_i + \Phi_i$. For systems with non-ideal constraints,

*Prikl.Matem.Mekhan.,54,3,366-372,1990

$$\sum_{i=1}^{n} R_i \delta q_i = \tau \neq 0 \tag{1.4}$$

and this is true for any possible displacements.

We shall assume that the manifold of admissible states of the system generated by the servo-constraints only is expressed as

$$q_i = a_i^* (q_\mu, t), \quad q_i^{\cdot} = b_i^* (q_\mu, p_\nu, t) \\ (a_i^* \in C_2;\ b_i^* \in C_1;\ \mu = 1, 2, \ldots, k;\ \nu = 1, 2, \ldots, m) \tag{1.5}$$

and that the constraints of the first kind are ideal. It then follows from (1.4) that

$$\sum_{i=1}^{n} \Phi_i \delta q_i = \tau \tag{1.6}$$

for any possible displacement, and the reaction forces of the servo-constraints can be resolved uniquely into components Φ_i^n and Φ_i^τ such that the left-hand side of the equality vanishes for Φ_i^n, while the quantities $\Phi_i^\tau \delta t$ are admissible displacements. In this situation

$$\Phi_i^n = \sum_{\alpha=c+1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} + \sum_{\beta=d+1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}}, \quad \Phi_i^\tau = \sum_{\nu=1}^{m} u_\nu \frac{\partial b_i^*}{\partial p_\nu} \tag{1.7}$$

where λ_α and μ_β are undetermined Lagrange multipliers, and u_ν certain coefficients of proportionality.

The motion of the system is described by the equations

$$\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = Q_i + N_i + \Phi_i \tag{1.8}$$

in which the kinetic energy T is constituted without allowing for the constraints (1.1) and (1.2), and the generalized reaction forces of the constraints have the following structure:

$$N_i = \sum_{\alpha=1}^{c} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} + \sum_{\beta=1}^{d} \mu_\beta \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}} \\ \Phi_i = \sum_{\alpha=c+1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} + \sum_{\beta=d+1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}} + \sum_{\nu=1}^{m} u_\nu \frac{\partial b_i^*}{\partial p_\nu} \tag{1.9}$$

Along with the general Eqs.(1.8), in any consideration of systems with servo-constraints the equations of motion must also reflect the specific physical realization of the system. Let us consider one such system, which occurs not infrequently in applications /1/.

2. The possibility of interpreting conditional constraints (servo-constraints) as real constraints is known as the principle of reduction of conditional to real constraints /4/. The use of this principle involves incorporating in the system certain conditions, dictating that the reactions of the constraints equal zero. Through a specific example, it was shown in /8/ that the application of this principle to the solution of Béghin's problem leads to a contradictory conclusion. Below we shall clarify the reasons for this contradiction and establish conditions under which the theories in /1, 4/ can be reconciled.

Following Béghin, we shall assume that a system subject to $e + f\ (e = a - c,\ f = b - d)$ servo-constraint relations can be split into two parts, Σ and Σ_1, such that Σ is not subject to any reactions of constraints of the second kind other than the reactions of Σ_1. Let us assume that the position of system Σ_1, which is subject to the servo-constraint reaction forces Φ_i, is defined by coordinates $q_{l+j}\ (j = 1, 2, \ldots, n - l)$ from the complete set $q_i\ (i = 1, 2, \ldots, n)$. Then the motion of system Σ will be described by the first l equations of system (1.8), in which we must substitute $\Phi_i = 0\ (i = 1, 2, \ldots, l)$. Associated with these equations, as in the case of /1/, are the constraints (1.1) and (1.2). The problem of the motion of the system is well defined if the number of servo-constraints equals the number $n - l$ of parameters determining the position of system Σ_1.

To determine the servo-constraint reaction forces $\Phi_i\ (i = l + 1, l + 2, \ldots, n)$ one uses the remaining $n - l$ equations of system (1.8). These are considered together with the relations

$$\sum_{\alpha=c+1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} + \sum_{\beta=d+1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}} + \sum_{\nu=1}^{m} u_\nu \frac{\partial b_i^*}{\partial p_\nu} = 0 \tag{2.1}$$

which follow from the condition that the generalized reaction forces of the servo-constraints, referred to the coordinates q_i $(i = 1, 2, \ldots, l)$, must vanish.

Consequently, in the case of the systems under consideration the principle of reduction of conditional to real constraints is equivalent to the usual method of incorporating the constraints and subsequently adding conditions (2.1), according to which the reaction forces of the servo-constraints, referred to system Σ, must vanish.

We shall show that the proposed method for investigating systems with servo-constraints enables one to find the conditions under which the theories of /1, 4/ can be reconciled. Indeed, the kinetic energy in Eqs.(1.8) does not involve the constraints (1.1) and (1.2). It may therefore be expressed as

$$T = T(\Sigma) + T(\Sigma_1)$$

where $T(\Sigma)$ and $T(\Sigma_1)$ are the kinetic energies of systems Σ and Σ_1, respectively. Since only $T(\Sigma)$ depends on the coordinates q_i $(i = 1, 2, \ldots, l)$. Eqs.(1.8) yield a system of equations

$$\frac{d}{dt}\frac{\partial T(\Sigma)}{\partial q_i^{\cdot}} - \frac{\partial T(\Sigma)}{\partial q_i} = Q_i + N_i \quad (i = 1, 2, \ldots, l) \tag{2.2}$$

to which one adds relations (1.1) and (1.2), if one is not interested in the reaction forces of the servo-constraints. Introducing the notation $q_{l+\chi} = v_\chi$ $(\chi = 1, 2, \ldots, n - l)$ and writing the equations of the constraints as

$$f_\alpha (q_i, v_\chi, t) = 0 \quad (\alpha = 1, 2, \ldots, a; \quad i = 1, 2, \ldots, l) \tag{2.3}$$
$$\varphi_\beta (q_i, q_i^{\cdot}, v_\chi, v_\chi^{\cdot}, t) = 0 \quad (\beta = 1, 2, \ldots, b; \ \chi = 1, 2, \ldots, n - l)$$

we obtain parametric constraints, which differ from those considered in /10/ in that the functions φ_β also involve the derivatives of the parameters $v_\chi^{\cdot}$ $(\chi = 1, 2, \ldots, n - l)$.

Eqs.(2.2), together with the constraints (2.3), comprise $a + b + l$ equations in $c + d + n$ unknowns. The problem will be well-defined if the number of parameters v_χ is the same as the number $e + f$ of servo-constraints. To determine the reaction forces of the servo-constraints applied to system Σ_1 one uses the remaining equations of (1.8), considered together with relations (2.1).

Remark 1. Kirgetov /10/, considering a controllable system Σ with constraints (2.3), postulates the D'Alembert-Lagrange principle in the form

$$\sum_{i=1}^{l} \left[\frac{d}{dt} \frac{\partial T(\Sigma)}{\partial q_i^{\cdot}} - \frac{\partial T(\Sigma)}{\partial q_i} - Q_i \right] \delta q_i = 0 \tag{2.4}$$

valid for any possible displacement determined by the conditions

$$\sum_{i=1}^{l} \frac{\partial f_\alpha}{\partial q_i} \delta q_i = 0, \quad \sum_{i=1}^{l} \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}} \delta q_i = 0 \quad (\alpha = 1, 2, \ldots, c; \beta = 1, 2, \ldots, d) \tag{2.5}$$

This makes it possible to derive the equations of motion in the form of (2.2). The latter are considered in conjunction with constraints (2.3), among which there are e geometric and f kinematic servo-constraints. Since these constraints are reduced in /4/ to real constraints, one must consider conditions (2.5) together with the relations

$$\sum_{i=1}^{l} \frac{\partial f_\alpha}{\partial q_i} \delta q_i = 0, \quad \sum_{i=1}^{l} \frac{\partial \varphi_\beta}{\partial q_i^{\cdot}} \delta q_i = 0 \tag{2.6}$$
$$(\alpha = c + 1, c + 2, \ldots, a; \quad \beta = d + 1, d + 2, \ldots, l)$$

The equations of motion in the form (2.2) can be derived from (2.4), on the assumption that conditions (2.5) and (2.6) are satisfied, if and only if conditions (2.1) are satisfied.

From the point of view of Béghin's theory, the total work (in the entire system) of the servo-constraint reaction forces over the possible displacements vanishes if one puts $\delta q_i = 0 \ (i = l+1, l+2, \ldots, n)$ in (1.6). When that is done, Eqs.(2.2) with conditions (1.1) and (1.2) are derivable from the general theory if and only if conditions (2.1) are satisfied.

It follows that conditions (2.1) are necessary and sufficient for the conclusions of the theories of /1/ and /4/ to be identical.

Remark 2. In the absence of parametric contact constraints in Eqs.(2.2), one must put $N_i = 0$ and add $e + f$ servo-constraints.

3. Along with the equations of the servo-constraints in systems (1.1) and (1.2), we must also consider the relations

$$f_{c+\gamma}(q_i, t) = \eta_\gamma \quad (\gamma = 1, 2, \ldots, e)$$
$$\varphi_{d+\rho}(q_i, q_i^{\cdot}, t) = \zeta_\rho \quad (\rho = 1, 2, \ldots, f) \tag{3.1}$$

where η_γ and ζ_ρ are parameters characterizing the continuous disengagement of the system from the geometric and kinematic constraints. With this parametric disengagement /9/, the deviations in the system are essentially represented by the left-hand sides of the servo-constraints, evaluated for real motion /12/, and instead of (1.5) one obtains the following representation for the manifold of admissible states:

$$q_i = A_i^* (q_\mu, \eta_\gamma, t) \quad (A_i^* \in C_1)$$
$$q_i^{\cdot} = B_i^* (q_\mu, \eta_\gamma, p_\nu, \zeta_\rho, \eta_\gamma^{\cdot}, t) \quad (B_i^* \in C_1) \tag{3.2}$$

where substituting $\eta_\gamma = \eta_\gamma^{\cdot} = \zeta_\rho = 0$ yields Eqs.(1.5), which describe the manifold of admissible states of the system prior to disengagement.

To incorporate the c geometric and d kinematic constraints of the first kind in systems (1.1) and (1.2), we transform them to the variables defining the manifold (3.2) and assume that the geometric constraints thus obtained may be solved for the variables $q_{p+1}, q_{p+2}, \ldots, q_k$ and the kinematic constraints for the variables $p_{r+1}, p_{r+2}, \ldots, p_m$.

The manifold of admissible states of the system is defined by equations

$$q_i = A_i (q_j, \eta_\gamma, t)$$
$$q_i^{\cdot} = B_i (q_j, \eta_\gamma, p_s, \zeta_\rho, \eta_\gamma^{\cdot}, t) \tag{3.3}$$

in which case the variations of the coordinates δq_i are expressed in terms of arbitrary variations $\delta\pi_s, \delta\sigma_\rho, \delta\eta_\gamma$ as follows:

$$\delta q_i = \sum_{s=1}^{r} \frac{\partial B_i}{\partial p_s} \delta\pi_s + \sum_{\rho=1}^{f} \frac{\partial B_i}{\partial \zeta_\rho} \delta\sigma_\rho + \sum_{\gamma=1}^{e} \frac{\partial B_i}{\partial \eta_\gamma} \delta\eta_\gamma \tag{3.4}$$

As we know /8, the D'Alembert-Lagrange principle for systems with servo-constraints can be written as

$$\sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} - Q_i - \Phi_i \right) \delta q_i = 0 \tag{3.5}$$

where the variations are related by conditions admissible by constraints of the first kind.

If allowance is made for the relations

$$\sum_{i=1}^{n} \frac{\partial f_{c+\gamma}}{\partial q_i} \delta q_i = \delta\eta_\gamma, \quad \sum_{i=1}^{n} \frac{\partial \varphi_{d+\rho}}{\partial q_i^{\cdot}} \delta q_i = \delta\sigma_\rho$$

then the total work of the servo-constraint reaction forces may be reduced to a form enabling us to write Eq.(3.5) as follows:

$$\sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} - Q_i - \Phi_i^{\tau} \right) \delta q_i = \sum_{\alpha=1}^{e} \lambda_{c+\alpha} \delta\eta_\alpha + \sum_{\beta=1}^{f} \mu_{d+\beta} \delta\sigma_\beta$$

Replacing δq_i by their values (3.4), we obtain a system of differential equations which, by introducing the acceleration energy S, evaluated in accordance with (3.3), can be transformed to

$$\frac{\partial S}{\partial p_s^{\cdot}} = Q_s^p + \Phi_s^p, \quad Q_s^p = \sum_{i=1}^n Q_i \frac{\partial B_i}{\partial p_s}, \quad \Phi_s^p = \sum_{i=1}^n \Phi_i^\tau \frac{\partial B_i}{\partial p_s} \tag{3.6}$$

$$\frac{\partial S}{\partial \eta_\alpha^{\cdot\cdot}} = Q_\alpha^\eta + \Phi_\alpha^\eta + \lambda_{c+\alpha}, \quad Q_\alpha^\eta = \sum_{i=1}^n Q_i \frac{\partial B_i}{\partial \eta_\alpha^{\cdot}}, \quad \Phi_\alpha^\eta = \sum_{i=1}^n \Phi_i^\tau \frac{\partial B_i}{\partial \eta_\alpha^{\cdot}}$$

$$\frac{\partial S}{\partial \zeta_\beta^{\cdot}} = Q_\beta^\zeta + \Phi_\beta^\zeta + \mu_{d+\beta}, \quad Q_\beta^\zeta = \sum_{i=1}^n Q_i \frac{\partial B_i}{\partial \zeta_\beta}, \quad \Phi_\beta^\zeta = \sum_{i=1}^n \Phi_i^\tau \frac{\partial B_i}{\partial \zeta_\beta}$$

Together with the kinematic relations (3.3), these equations can be used to determine the unknowns $p_s, q_i, \eta_\alpha, \zeta_\beta$. The control parameters in these equations are $\lambda_{c+1}, \lambda_{c+2}, \ldots, \lambda_a; \mu_{d+1}, \mu_{d+2}, \ldots, \mu_b; u_1, u_2, \ldots, u_m$.

For the systems considered in Sect.2, these equations must be taken together with conditions (2.1).

Remark 3. Introducing the notation

$$\eta_\gamma^{\cdot} = y_\gamma, \quad \zeta_\rho = y_{e+\rho}, \quad \eta_\gamma = y_{q+\gamma}$$

$$\eta^{\cdot\cdot}_\gamma = V_\gamma, \quad \zeta_\rho^{\cdot} = V_{e+\rho} \; (q = e + f)$$

we obtain a system /11/

$$y^{\cdot} = Ay + BV \tag{3.7}$$

This system, which describes the deviation of the motion from the servo-constraints, is completely controllable /13/, and one can always find controls of the form $V = V(y), V(0) = 0$, which stabilize the trivial solution of the equations

$$y^{\cdot} = Ay + BV(y), \quad y(0) = y^\circ$$

Considering this system in conjunction with Eqs.(3.6), one can determine servo-constraint reaction forces that stabilize the motion relative to the manifold defined by the servo-constraints, as well as equations whose limiting values correspond to the motion of the system dictated by these conditions.

4. As examples of the application of our method to the investigation of systems with servo-constraints, we consider some problems of Béghin.

Example 1. Retaining all the notation of /1, Sect.17/, consider the motion of a plate Σ attached by a hinge to a circular disk Σ_1.

Confining our attention first to the case in which the servo-constraints

$$\alpha - \beta - \pi/2 = 0 \tag{4.1}$$

are satisfied exactly, and assuming that the variations of the coordinates are such that and that all conditions of Remark 2 are satisfied, we have

$$T = T(\Sigma) + T(\Sigma_1) = \frac{1}{2}\{M[R^2\alpha^{\cdot 2} + (b^2 + k^2)\beta^{\cdot 2} + 2bR\alpha^{\cdot}\beta^{\cdot}\cos(\alpha - \beta)] + I_1\alpha^{\cdot 2}\}, \quad Q_\alpha = -RF\sin\alpha$$

$$Q_\beta = -aF\sin\beta, \quad \Phi_\alpha^n = -\Phi_\beta^n = \lambda, \quad \Phi_\alpha^\tau = \Phi_\beta^\tau = u$$

The equations of motion (1.8), together with condition (2.1), which imply $\lambda = u$, lead to the system

$$M[(b^2 + k^2)\beta^{\cdot\cdot} - bk\beta^{\cdot 2}] + aF\sin\beta = 0$$

$$[M(b^2 + k^2 + R^2) + I_1]\beta^{\cdot\cdot} + F(a\sin\beta + R\cos\beta) = 2u \tag{4.2}$$

Remark 4. In the framework of the theory of /10/, the general equation of the dynamics of the system involves only prescribed forces. Therefore, the derivation of Eqs.(7.5) in /4/, based on incorporating in the prescribed force Q_α a parameter u characterizing the reaction, lacks a rigorous basis. Application of the theory of /4/ to solve this problem involves the introduction of a parametric contact constraint, such as an equation expressing the condition that a servo-motor /1/ is coupled with the disk Σ_1.

Let us assume now that the initial conditions of the system are incompatible with Eq.(4.1) and it is required to solve the problem of stabilizing the motion relative to this manifold.

Using the relations

$$\alpha = x + \eta + \pi/2, \quad \beta = x$$

which identically satisfy the condition

$$\alpha - \beta - \pi/2 = \eta$$

we introduce independent coordinates x and η, in terms of which we express the general equation of the dynamics of the system:

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \alpha^{\cdot}} - \frac{\partial T}{\partial \alpha} - Q_\alpha - \Phi_\alpha^{\tau}\right)\delta\alpha + \left(\frac{d}{dt}\frac{\partial T}{\partial \beta^{\cdot}} - \frac{\partial T}{\partial \beta} - Q_\beta - \Phi_\beta^{\tau}\right)\delta\beta = \lambda\delta\eta$$

The result is a system of equations which, in conjunction with the equation

$$\eta^{\cdot\cdot} = V(\eta, \eta^{\cdot}), \quad V(0, 0) = 0, \quad \eta(0) = \eta^{\circ}, \quad \eta^{\cdot}(0) = \eta^{\cdot\circ} \tag{4.3}$$

whose trivial solution is asymptotically stable, can be written as

$$[M(b^2 + k^2 + R^2 - 2bR\sin\gamma) + I_1]\beta^{\cdot\cdot} + [MR(R - b\sin\gamma) + I_1]V(\eta, \eta^{\cdot}) - MbR\eta^{\cdot}(\eta^{\cdot} + 2\beta^{\cdot})\cos\eta + F[a\sin\beta + R\cos(\beta + \eta)] = 2u \tag{4.4}$$

$$[MR(R - b\sin\eta) + I_1]\beta^{\cdot\cdot} + (MR^2 + I_1)V(\eta, \eta^{\cdot}) + MbR\beta^{\cdot 2}\cos\eta + RF\cos(\beta + \eta) = 2u$$

Eqs.(4.3) and (4.4) enable one to determine the motion of the system, as well as a servo-constraint reaction force Φ_α which stabilizes the motion relative to the manifold defined by the servo-constraint (4.1). In that case letting $\eta \to 0$ in Eqs.(4.4) we obtain a limiting system which can be reduced to the form of (4.2).

Example 2. Consider the problem of a homogeneous sphere of radius R sliding without friction over a material plane P. Retaining all the notation of /1, Sect.21/, let us assume that the plane P, upon which the reaction forces of the servo-constraints are acting, is of mass m, and the Euler kinematic equations are given by

$$p = \theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\psi\sin\theta$$
$$q = \theta^{\cdot}\sin\psi - \varphi^{\cdot}\cos\psi\sin\theta$$
$$r = \psi^{\cdot} + \varphi^{\cdot}\cos\theta$$

The system, which is moving subject to constraints of the first kind

$$\xi^{\cdot} - u^{\cdot} - R(\theta^{\cdot}\sin\psi + \varphi^{\cdot}\cos\psi\sin\theta) = 0$$
$$\eta^{\cdot} - v^{\cdot} + R(\theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\psi\sin\theta) = 0 \tag{4.5}$$

must be subjected to the servo-constraints

$$\xi^{\cdot} + \omega\eta = 0, \quad \eta^{\cdot} - \omega\xi = 0 \tag{4.6}$$

Let us construct the equations of motion and determine the structure of the servo-constraint reaction forces that stabilize the motion relative to the manifold defined by (4.6).

The acceleration energy of the system is

$$S = S(\Sigma) + S(\Sigma_1) = {}^1/_2 M(\xi^{\cdot\cdot 2} + \eta^{\cdot\cdot 2}) + {}^1/_2 A(p^{\cdot 2} + q^{\cdot 2} + r^{\cdot 2}) + {}^1/_2 m(u^{\cdot\cdot 2} + v^{\cdot\cdot 2})$$
$$(A = {}^2/_5 MR^2)$$

Using (1.8) to determine the components of the servo-constraint reaction forces and noting that by (2.1) the reaction forces referred to the coordinates of system Σ must vanish, we deduce that the only non-zero components are $\Phi_u = \Phi_u^{\tau}$, $\Phi_v = \Phi_v^{\tau}$.

Considering the equations

$$\xi^{\cdot} + \omega\eta = \zeta_1, \quad \eta^{\cdot} - \omega\xi = \zeta_2 \tag{4.7}$$

we use the relations

$$\xi^{\cdot} = \zeta_1 - \omega\eta, \quad \eta^{\cdot} = \zeta_2 + \omega\xi$$
$$\theta^{\cdot} = R^{-1}[(p_2 - \zeta_2 - \omega\xi)\cos\psi + (\zeta_1 - p_1 - \omega\eta)\sin\psi]$$
$$\psi^{\cdot} = R^{-1}\operatorname{ctg}\theta[(p_2 - \zeta_2 - \omega\xi)\sin\psi + (\zeta_1 - p_1 - \omega\eta)\cos\psi] + r \tag{4.8}$$
$$\varphi^{\cdot} = (R\sin\theta)^{-1}[(p_2 - \zeta_2 - \omega\xi)\sin\psi + (p_1 - \zeta_1 + \omega\eta)\cos\psi]$$
$$u^{\cdot} = p_1, \quad v^{\cdot} = p_2$$

which identically satisfy conditions (4.5) and (4.7), to introduce velocity parameters $\zeta_1, \zeta_2, p_1, p_2, r$. Transforming the acceleration energy to these variables, we obtain

$$S = {}^1/_2 M\langle[\zeta_1^{\cdot} - \omega(\zeta_2 + \omega\xi)]^2 + [\zeta_2^{\cdot} + \omega(\zeta_1 - \omega\eta)]^2\} + {}^2/_5\{[\zeta_1^{\cdot} - p_1^{\cdot} - \omega(\zeta_2 + \omega\xi)]^2 + [p_2^{\cdot} - \zeta_2^{\cdot} - \omega(\zeta_1 - \omega\eta)]^2 + R^2 r^{\cdot 2}\}\rangle + {}^1/_2 m(p_1^{\cdot 2} + p_2^{\cdot 2})$$

Writing the equations of motion in the form (3.6) and adding the equations

$$\zeta_1^{\cdot} = V_1(\zeta_1), \quad V_1(0) = 0, \quad \zeta_1(0) = \zeta_1^{\circ}$$
$$\zeta_2^{\cdot} = V_2(\zeta_2), \quad V_2(0) = 0, \quad \zeta_2(0) = \zeta_2^{\circ} \tag{4.9}$$

whose trivial solution is asymptotically stable, we obtain the system

$$p_1^{\cdot} - {}^7/_2\,[V_1(\zeta_1) - \omega(\zeta_2 + \omega\xi)] = 0$$
$$p_2^{\cdot} - {}^7/_2\,[V_2(\zeta_2) + \omega(\zeta_1 - \omega\eta)] = 0 \tag{4.10}$$
$$(M + {}^7/_2 m)\,[V_1(\zeta_1) - \omega(\zeta_2 + \omega\xi)] = \Phi_u^{\tau}$$
$$(M + {}^7/_2 m)\,[V_2(\zeta_2) + \omega(\zeta_1 - \omega\eta)] = \Phi_v^{\tau}$$

Eqs.(4.9) and (4.10), in conjunction with the kinematic relations (4.8), determine the motion of the system and the reaction forces of the servo-constraints. Letting $\zeta_1 \to 0$, $\zeta_2 \to 0$ in Eqs.(4.10) and (4.8) we obtain a limiting system corresponding to the satisfaction of the servo-constraint relations (4.6).

REFERENCES

1. BÉGHIN H., Etude Théorique des Compas Gyrostatiques, Nauka, Moscow, 1967.
2. PAINLEVE P., Lecons sur le Frottement, Gostekhizdat, Moscow, 1954.
3. RUMYANTSEV V.V., On systems with friction. Prikl. Mat. Mekh., 25, 6, 1961.
4. KIRGETOV V.I., On the motion of controllable mechanical systems with conditional constraints (servo-constraints). Prikl. Mat. Mekh., 31, 3, 1967.
5. POZHARITSKII G.K., On the equations of motion for systems with non-ideal constraints. Prikl. Mat. Mekh., 24, 3, 1960.
6. NUGMANOVA SH.S., On the equations of motion of controllable systems. Trudy Kazan. Aviats. Inst., 27, 1953.
7. RUMYANTSEV V.V., On the motion of certain systems with non-ideal constraints. Vestnik Moskov. Gos. Univ., Ser. 1. Mat., Mekh., 5, 1961.
8. RUMYANTSEV V.V., On the motion of controllable mechanical systems. Prikl. Mat. Mekh., 40, 5, 1976.
9. CHETAYEV N.G., Stability of Motion. Papers in Analytical Mechanics, Izd. Akad. Nauk SSSR, Moscow, 1962.
10. KIRGETOV V.I., On kinematically controllable mechanical systems. Prikl. Mat. Mekh., 28, 1, 1964.
11. AZIZOV A.G., On the motion of a certain controllable system of variable mass, Prikl. Mat. Mekh., 50, 4, 1986.
12. GALIULLIN A.S., MUKHAMETZYANOV I.A., MUKHARLYAMOV R.G. and FURASOV V.D., Construction of Systems of Programmed Motion, Nauka, Moscow, 1971.
13. KRASOVSKII N.N., Theory of Control of Motion, Nauka, Moscow, 1968.

Translated by D.L.